TAXES, BANKRUPTCY, AND THE TALMUD

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This discussion continues and augments ideas from Consortium *95, Fall/Winter 2008, pp. 3–7. It also reviews concepts from that article in order to be self-contained.*

What is a fair way for a democracy to tax its citizens? It might seem at first glance that the mathematics associated with this question is merely a matter of constructing charts and tables to display the arithmetic of different tax rates that could be chosen. Rather surprisingly, this complex political question provides an interesting glimpse at how mathematics grows and evolves, and how mathematics created for one situation can often be exported in a profitable way to get dramatic insights into other domains.

The story begins astonishingly enough with the Babylonian Talmud! The Talmud is a collection of commentaries on the (Hebrew) Bible that originated as an oral tradition and eventually was codified into written form. In the book of Leviticus, many rules and regulations for the way to lead a "good life" are set out. However, translating the laws laid down in that book to day-to-day issues plagued those who tried to live in accordance with the Bible. One theme of the Talmud is the attempt by the rabbis who developed commentaries on the Bible to make sense of conflicting goals and objectives when fairness is an issue. Often the discussion takes the form of a specific situation or problem that needs "adjudication." Here, summarized, are some problems that are related to the idea of a "bankruptcy." One has claimants whose claims are not in doubt, but the "estate" or "remaining assets" in the case of a bankruptcy do not cover all the claims that are valid.

The first and the last entries of **Table 1** make some sense. If the amount to pay off a bankruptcy is relatively small compared with the total amount being claimed, it seems entirely reasonable to give each claimant an equal share. Some would argue that this way to solve bankruptcies—equal shares for all claimants—makes sense as long as the amount given to a claimant is never more than what is claimed. This fairness method now goes by the name of the Maimonides (Gain) Method. The name originates from the medieval philosopher Moses Maimonides (1135–1204) who analyzed this kind of fairness principle. Maimonides also realized that trying to equalize gains after a settlement often still left significant losses, and these losses could be unequal losses for some of the claimants. In some ways, this does not seem fair. So, Maimonides also discussed what has come to be known as the Maimonides Loss Method. This solution approach equalizes the losses of the claimants as much as possible but never requires that a claimant "subsidize" the estate (that is, add more money to the estate out of his/her own pocket). The method of "pure" or complete equalization of loss

Estate/Claim	100 (Claimant 1)	200 (Claimant 2)	300 (Claimant 3)
100	33-1/3	33-1/3	33-1/3
200	50	75	75
300	50	100	150

TABLE 1.

requires claimants to subsidize the original size of the estate with money out of their own pockets so that after subsidization (if necessary) the claimants all suffer the same loss.

From the discussion above one sees the possibility of four different methods:

1. Equality of gain. Give claimants equal amounts, even if this means they get more than they claimed.

2. Maimonides gain. Give claimants as equal amounts as possible, but never more than what they claimed.

3. Equality of loss. Equalize losses for the claimants, even if a claimant has to add money to the estate to achieve loss equality.

4. Maimonides loss. Equalize loss as much as possible without claimants' reaching into their own pockets to help make the losses totally equal.

We will see that these four methods are relevant to understanding the ideas about bankruptcy problems in the Talmud and how the scholars who developed the Talmud thought about these questions.

Let us now return to the last line of Table 1. Clearly, this is not an equal division of the estate from the point of gains or loss. What is going on? The thinking behind this solution is that the claimants have different stakes in the estate related to the size of their claims. People with larger portions of the total amount claimed deserve more of the estate from this point of view. This method is known as proportionality of gain. We begin by computing the sum of the claims for all the claimants, and then for each claimant the percentage of the total claim being made by that claimant. You can do the calculations for yourself and verify that the third line of Table 1 arises by using the "proportional gain" method. For example, the total claims in looking at line 3 of Table 1 amount to 600. Thus, the first claimant's portion of this is 100/600 = 1/6. If we give Claimant 1 (the one with a claim of 100) 1/6 of 300 (the size of the estate), this claimant would get 50, as recorded in line 3. The other entries in that row are computed in a similar manner.

What about proportional losses? We have noted that gains and losses for a "method" may not always work the same way. For line 3, we have total claims of 600 and an estate of 300. Thus, the amount that collectively must be shared in losses is exactly the same in this example; it is also 300, the same numerical amount as the gains that could be distributed to the claimants, so assigning losses proportional to claims gives the same result as assigning gains proportional to claims. Here this happened by "accident," but you should check for yourself that, in general, the amounts given to claimants will be the same when the claims are resolved by distributing gains or losses proportional to claims. This turns out to be an exercise in basic algebra. It is also true that in United States law, bankruptcies are typically solved by applying the proportionality method, though sometimes the claims are divided into priority classes, with high priority classes being treated in a different way from those with lower priority. It is also worth noting that the proportional method can also be thought of as paying off each claimant with a certain fixed number of cents per dollar of original claim. (Here we assume the unit of the claims is dollars.) For example, in this case each claimant (when the estate is $300) is given 50 cents per dollar. Thus, the second claimant with a 200-unit claim would be given $100.

So, this brings us to the "mysterious" second line of Table 1! These numbers do not correspond to either the proportional method or one of the four methods (equality of gain/loss, Maimonides gain/loss) we treated earlier. What is the thinking behind these numbers?

At this point, we are in the same position as the biblical scholars who were studying the thinking behind questions that arose in the Babylonian Talmud. Based on discussions of equity and "bankruptcy" questions in Talmud texts, they were unable to understand the numbers in line 2 of Table 1. There was one other approach to these questions that such scholars had at their disposal that we have not yet discussed. This idea, though quite easy to understand in practice and principle, intriguingly had not been part of the thinking in dealing with equity issues in recent years. It goes under the colorful name of the "contest garment rule."

Here is the situation, which may not quite seem to fit into the estate or bankruptcy mold, since garments are not usually taken as being divisible assets (from H. P. Young, *Equity: In Theory and Practice*):

> *Two hold a garment: one claims it all, and the other claims half. What is an equitable division of the garment?*

The solution given for this problem is 3/4 and 1/4. This is neither the proportional solution nor equality of gain, though it does correspond to an equalizing loss solution. However, the reasoning behind this does not involve equalization of loss! To explain the ideas, we will look at a problem that is more in the spirit of the problems in Table 1.

In **Table 2**, there are two claimants asking $100 and $150 from an estate that does not have enough money to pay off the claims fully.

Estate/Claim	$100 (Claimant 1)	$150 (Claimant 2)
$130	50	80
$220	85	135

TABLE 2.

How does the “contested garment rule” (sometimes called the Talmudic rule) work for the two cases in the table?

Let us refer to the person who is distributing the money as the “judge.” We assume the judge is sensible, fair, and will not get paid from the estate for his/her services. Claimant 2, looking at the claims requested, approaches the judge with the following argument. There are only two people claiming this money, and you are going to distribute $130. Claimant 1 is only asking for $100, hence, of the $130 you have $30 *must* be assigned to me! The judge agrees, and notes that since Claimant 2 is claiming more than the estate, Claimant 1 cannot make a similar case. Thus, the judge gives $30 to Claimant 2, and decides that the two claimants have equal claims on the remaining amount. (Thus, in this case we would give Claimant 2 $80 ($30 + $50) while Claimant 1 gets $50.) In a situation like this, we say that Claimant 2 had an uncontested claim against Claimant 1, and that the amount of this claim was $50.

Is it possible for both claimants to have an uncontested claim against each other? Yes, this is possible, as illustrated in the second line of Table 2. Claimant 1 has an uncontested claim of $70 against Claimant 2, because Claimant 2 is claiming $150 of the $220 the judge has to distribute. Similarly, Claimant 2 has a $120 claim against Claimant 1. When the judge adds these two amounts, $70 and $120, and subtracts from the $220 that there is to distribute, there is $30 left. Distributing half of this “mutually contested” claim equally with what the claimants have already been “assigned” using uncontested claims means that $85 ($70 + $15) is given to Claimant 1, and $135 ($120 + $15) to Claimant 2. Note that in the first line of Table 2 this solution does not equalize the losses to the claimants, while in the second line it does. It turns out there is a theorem lurking here: When the size of the estate is greater than that of the individual claims, the solution using the “contested garment rule” coincides with the method that equalizes the losses of the claimants.

So, what does this discussion have to do with the puzzle in Table 1’s second line? Again, the purpose of our discussion here is to acquaint you with the “facts” known at the time the puzzle was being studied. The story begins with an important paper by the political scientist Barry O’Neill, from 1982, entitled “A Problem of Rights Arbitration in the Talmud.” O’Neill’s paper constitutes one of the many interactions between people trained outside of mathematics but interested in mathematics, and the mathematics community.

The following account is pieced together from the writings of R. J. Aumann, an American-educated mathematician who eventually took up game theory as his life’s work, and who won the Nobel Memorial Prize for his work in the theory of games. Aumann and his Hebrew University colleague, Michael Maschler, wrote a very influential paper on this topic. Aumann relates that in 1982, while visiting at Stanford, he saw a preprint of O’Neill’s paper. He sent a copy of the paper to his son Shlomo, a biblical scholar (who not long thereafter died while on reserve duty in “Operation Peace for Galilee”). Shlomo Aumann suggested a specific place in the Talmud for his father to consider (Ketuvot 93a—in essence our Table 1). Aumann writes, “I could not make sense of it.” On his return to Israel, he worked with Michael Maschler to make sense of Table 1. Initially they were unsuccessful.

Aumann writes:

> *Finally one of us said, let’s try the nucleolus; to which the other responded, come on, that’s crazy, the nucleolus is an extremely sophisticated notion of modern mathematical game theory, there’s no way that the sages of the Talmud could possibly have thought of it. What do you care, said the first; it will cost us just fifteen minutes of calculation. So we did the calculation, and the nine numbers came out precisely as in the Talmud!*

(I have seen another version of this story where the claim was made they used software to test the results that the nucleolus would give.)

So, in modern terms, it appears that the reason Table 1 has the numbers it does is that the scholars who produced the Babylonian Talmud had discovered the concept of the nucleolus of a special class of “games,” which today might be called bankruptcy games. The modern approach to the nucleolus involves ideas from linear programming and applies to more general kinds of games than bankruptcy games.

Suppose a subset of claimants X gets an amount A when the whole estate E is distributed to all the claimants. A solution method is called *consistent* if when A is distributed only to members of X, X’s members get the same amounts against their claims as when E was distributed to all the claimants. Consistent methods (think of obeying being consistent as obeying a particular “axiom”) come up not only in bankruptcy problems, but also in other kinds of fair division questions. For example, the Huntington–Hill method of apportioning the U.S. House of Representatives, which is the method that will be used to apportion the House of Representatives after the 2010 census, is “consistent” in the sense that it treats pairs of states the same way as they are treated with the whole fifty states. The entries in Table 1 turn out to be generated by a consistent method, as discovered by Aumann and Maschler.

Aumann and Maschler discovered a new algorithm for finding the nucleolus of a bankruptcy game, which is one that could have been discovered and used by the scholars who worked on the Talmud. This algorithm grows out of the ideas of Maimonides gain and loss, as well as the contested garment rule. It also has some "psychological" significance. This comes from the idea that if one manages to get at least half of what one was claiming, this may seem important psychologically compared with the "bitterness" that would come if one got less than half of what one claimed.

Rather than discuss Aumann and Maschler's method in general, I will discuss it for a particular example. Also, I will consider it in the case of three players so that one can see how it generalizes the contested garment rule and how it satisfies the idea of consistency.

Consider three claimants, *A*, *B*, and *C*, as having (verified) claims of 50, 150 and 200 respectively, and an estate *E* worth $340.

The Aumann/Maschler algorithm works as follows. One first divides the claims in half. In this case, this means giving *A* $25, *B* $75, and *C* $100. Since, in this case, we have an estate of $340, and half the claims sum to $200, we can accomplish this goal. The method used to pay off this part of the claims is Maimonides gain; that is, we try to give each claimant as equal an amount as possible, but never more than what was claimed. So, in this case this can be done. The amount distributed at this phase is $200 (25 + 75 + 100). Since *E* was $340, we now have $140 additional to distribute to the three claimants. The way this remaining amount is distributed is by using the Maimonides loss method! To equalize loss as much as possible, we can first give *C* $25 since this will make *C*'s loss 75, and *B* (who so far has been given nothing) also has a loss of $75. At this stage, *A* has a loss of 25, and *B* and *C* each have a loss of $75. To continue to equalize loss, we could try to reduce both *B* and *C*'s losses down to the level of *A*'s loss, $25. Is there enough money left to do this? Yes, since after giving *C* $25, we now have $115 left to distribute. So, the judge can give *B* $50 and *C* $50, thereby reducing *B* and *C*'s losses to $25, which is now the same as *A*'s loss (*A* not yet having been given any money). So, the judge who had $140 to distribute in this stage now has $140 – $25 – $50 – $50 = $15 left. To continue to equalize loss, the judge can give *A*, *B*, *C* each $15/3 = $5.

So, what are the actual payments that *A*, *B*, and *C* get? *A* gets $25 in the first stage and $5 in the second for a total of $30; *B* gets $75 in the first stage and $55 ($50 + $5) in the second stage for a total of $130; *C* gets $100 in the first stage and $80 ($25 + $50 + $5) in the second stage for a total of $180.

Although the details of the assignment of money by this procedure are a bit complex, it only involves elementary work with arithmetic, and would have been possible in the Middle Ages. Note that this method does not give the values that Maimonides gain or Maimonides loss would for the claims involved. Furthermore, this solution is not the proportional solution either.

Now, let us verify that theoretically attractive property of the Talmudic Method—namely, consistency. What is involved here? Suppose we consider claimants *A* and *B* only. This solution gave them collectively $160. How would the Talmudic Method assign payoffs to *A* and *B* with claims of $50 and $150, if $160 were available? Proceeding according to the "contested garment rule," we compute *A* and *B*'s uncontested claims against each other. *A*'s uncontested claim against *B* is $10, while *B*'s uncontested claim against *A* is $110. Thus, we split the remaining $40 equally, giving *A* $10 + $20 = $30, and *B* $110 + $20 = $130, which is the amount of $160 that they were supposed to split. Note, however, this is exactly the split we found in the three-way problem! To verify that "consistency" holds, two further verifications are required. We have to see how claimants *B* and *C* with claims of $150 and $200 would share $310, and how claimants *A* and *C* with claims of $50 and $200 would share $210. Doing the calculations will show that this is indeed the case.

The work of O'Neill, Aumann and Maschler encouraged many people to have a second look at bankruptcy situations and the methods that could be used to "fairly" solve them. Just as Euclid tried to find the essential features of what today we call Euclidean Geometry, and Kenneth Arrow (the mathematically trained economist and Nobel Memorial Prize winner) axiomatized the desirable features of a social decision procedure (ways to amalgamate individual preferences into a societal choice), many scholars have tried to do the same for bankruptcy, notably William Thomson (Economics, University of Rochester).

In addition to looking at methods that obey "consistency," there are other "rules" or "axioms" to consider. For example, we have argued that the claimants are owed their claims independently and do not act in "concert" for advantage. For example, suppose Claimant *A* is a conglomerate with several businesses whose total claims could be thought of as two separate Claimants, *A*1 and *A*2, whose separate claims when they were added give *A*'s claim. Would the same amounts be given out when treating *A* as two claimants instead of as a single claimant? Similarly, we could look at two claimants who might "pretend" to be a single claimant because they would get more in total than when they were treated separately. Ideally, there are many "axioms" or rules that one would like bankruptcy solution methods to obey. There are now many such fairness rules and many theorems that show which sets of rules force one to use a particular method, or for which there is no method that obeys all of the rules.

Here are some other fairness rules that many people think make sense:

a. If two claimants have equal claims, they should get equal awards.

Comment: In apportionment problems this is often a hard condition to achieve or even justify. This is because in apportionment questions, the items being claimed are "indivisible," so when breaking a tie an arbitrary choice to give a whole valuable thing to one of the claimants must be made. However, for bankruptcy problems the claims are being made against the divisible quantity, money. If *A* were given more than *B*, though they had the same claims, why not take the difference between *A* and *B*'s awarded amounts, divide this in two and give each claimant this amount to equalize the claims?

b. Monotonicity with respect to estate size: If the amount in the estate *E* turns out to increase (perhaps an additional account in a bank that the judge was unaware of suddenly turns up), then the amounts that are given to the claimants should not decrease.

Comment: The reason why this rule is stated in the form that an awarded amount cannot go down is that it may remain the same. For the Maimonides gain rule, if a claimant is paid off completely with an estate of size *E*, then if *E* gets larger, the amount this claimant gets will not go up because the claimant has already been given all that is deserved. Thus, awarded amounts should not go down, but they may stay the same as an estate grows.

c. Monotonicity with respect to claim size: If the amount of the estate stays fixed but the claim of a particular claimant rises (perhaps because it is verified that more was owed to this claimant), then the amount given to this claimant should not decrease.

So what do tax issues have to do with bankruptcy? If one thinks of a society having different income classes as "claimants" and the size of the claim for each income class is the amount of money that that class could "potentially" be drawn on to pay in taxes, one has a "bankruptcy situation." The estate in this setting is the amount that government wishes to raise in income from taxes, and the amounts given to each income class correspond to the piece of the total amount of tax to be raised that each income class would give. There are many more people with low incomes than with high incomes, so, the amounts of money in each class reflect both the numbers of people in an income class and the amount of money those individuals have to pay towards tax.

The bankruptcy model is but one of many fairness and equity models (others include elections and voting, weighted voting, apportionment, fair division) that uses mathematical ideas to both increase our insight into economics and mathematics. These tools for understanding fairness environments are helping to further ways for us to create a fairer society.

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